(2-1-4) Limit Invloving Infinity:

Properties:

If

$$\lim_{x\to\infty} f_1(x) = L_1 \quad \text{and} \quad \lim_{x\to\infty} f_2(x) = L_2$$

and L_1 and L_2 are (finite) real numbers, then

1. *Sum Rule:* $\lim_{x\to\infty} [f_1(x) + f_2(x)] = L_1 + L_2$
2. *Difference Rule:* $\lim_{x\to\infty} [f_1(x) - f_2(x)] = L_1 - L_2$
3. *Product Rule:* $\lim_{x\to\infty} f_1(x) \cdot f_2(x) = L_1 \cdot L_2$
4. *Constant Multiple Rule:* $\lim_{x\to\infty} k \cdot f_1(x) = k \cdot L_1$ (any number k)
5. *Quotient Rule:* $\lim_{x\to\infty} \frac{f_1(x)}{f_2(x)} = \frac{L_1}{L_2}$ if $L_2 \neq 0$.

These properties hold for $x \to -\infty$ as well as $x \to \infty$.

EXAMPLE 5

$$\lim_{x\to\infty}\left(5 + \frac{1}{x}\right) = \lim_{x\to\infty} 5 + \lim_{x\to\infty}\frac{1}{x} \qquad \text{(Sum Rule)}$$

$$= 5 + 0 = 5. \qquad \text{(Known values)}$$

EXAMPLE 6

$$\lim_{x\to-\infty}\frac{4}{x^2} = \lim_{x\to-\infty} 4 \cdot \lim_{x\to-\infty}\frac{1}{x} \cdot \lim_{x\to-\infty}\frac{1}{x} \qquad \text{(Product Rule)}$$

$$= 4 \cdot 0 \cdot 0 = 0. \qquad \text{(Known values)}$$

EXAMPLE 7 $\lim_{x\to\infty}\frac{\sin x}{x} = 0$

Notice first that $\sin x$ lies between -1 and 1. Therefore, for all positive values of x,

$$-\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}.$$

2.3.1 THEOREM. *Let k be a real number.*

$$\lim_{x\to-\infty} k = k \qquad \lim_{x\to+\infty} k = k$$

$$\lim_{x\to-\infty} x = -\infty \qquad \lim_{x\to+\infty} x = +\infty$$

$$\lim_{x\to-\infty} \frac{1}{x} = 0 \qquad \lim_{x\to+\infty} \frac{1}{x} = 0$$

$$\lim_{x\to+\infty} x^n = +\infty, \quad n = 1, 2, 3, \ldots$$

$$\lim_{x\to-\infty} x^n = \begin{cases} -\infty, & n = 1, 3, 5, \ldots \\ +\infty, & n = 2, 4, 6, \ldots \end{cases}$$

Example 9:

$$\lim_{x\to+\infty} 2x^5 = +\infty, \qquad \lim_{x\to-\infty} 2x^5 = -\infty$$

$$\lim_{x\to+\infty} -7x^6 = -\infty, \qquad \lim_{x\to-\infty} -7x^6 = -\infty$$

The end behavior of a polynomials matches the end behavior of its highest degree term.

More precisely, if $c_n \neq 0$ *then*:

$$\lim_{x\to+\infty} (c_0 + c_1x + \cdots + c_nx^n) = \lim_{x\to+\infty} c_nx^n$$

$$\lim_{x\to-\infty} (c_0 + c_1x + \cdots + c_nx^n) = \lim_{x\to-\infty} c_nx^n$$

We can conclusion these results by factoring out the highest power of x from the polynomial and examining the limit of the factored expression.

Example10:

1- $\lim_{x\to\infty} (x^2 - 5x + 6) = \lim_{x\to\infty} x^2 = \infty$

2- $\lim_{x\to-\infty} (15x^7 - 3x^4 + x - 5) = \lim_{x\to-\infty} 15x^7 = -\infty$

(2-2) Continuity:

A function f is said to be continuous at x=c provided the following conditions are satisfied:

1. $f(c)$ is defined.
2. $\lim_{x \to c} f(x)$ exists.
3. $\lim_{x \to c} f(x) = f(c)$.

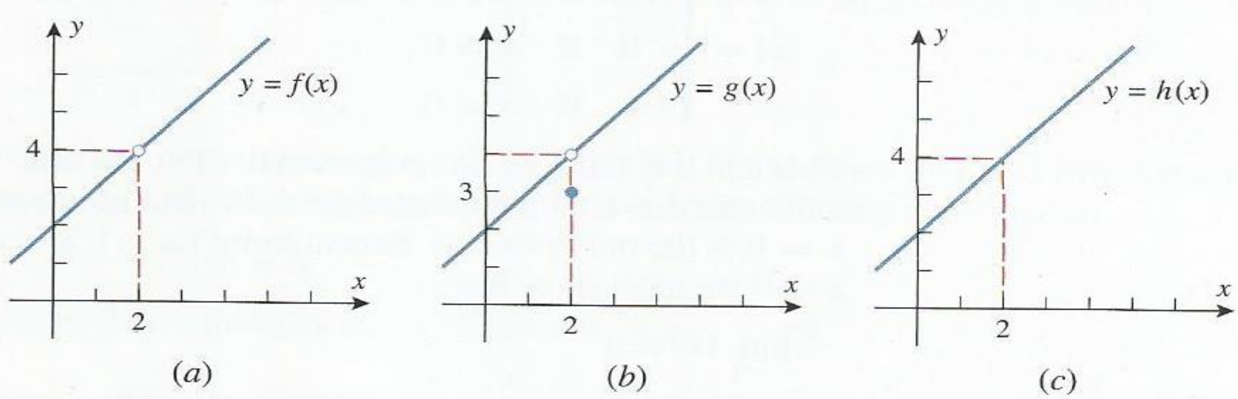
Example 1 Determine whether the following functions are continuous at $x = 2$.

$$f(x) = \frac{x^2-4}{x-2}, \qquad g(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x \neq 2 \\ 3, & x = 2, \end{cases} \qquad h(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x \neq 2 \\ 4, & x = 2 \end{cases}$$

Solution. In each case we must determine whether the limit of the function as $x \to 2$ is the same as the value of the function at $x = 2$. In all three cases the functions are identical, except at $x = 2$, and hence all three have the same limit at $x = 2$, namely

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} g(x) = \lim_{x \to 2} h(x) = \lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2}(x+2) = 4$$

The function f is undefined at $x = 2$, and hence is not continuous at $x = 2$ (Figure 2.5.4a). The function g is defined at $x = 2$, but its value there is $g(2) = 3$, which is not the same as the limit as x approaches 2; hence, g is also not continuous at $x = 2$ (Figure 2.5.4b). The value of the function h at $x = 2$ is $h(2) = 4$, which is the same as the limit as x approaches 2; hence, h is continuous at $x = 2$ (Figure 2.5.4c). (Note that the function h could have been written more simply as $h(x) = x + 2$, but we wrote it in piecewise form to emphasize its relationship to f and g.) ◀



Example 2 Show that $|x|$ is continuous everywhere (Figure 1.2.5).

Solution. We can write $|x|$ as

$$|x| = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

so $|x|$ is the same as the polynomial x on the interval $(0, +\infty)$ and is the same as the polynomial $-x$ on the interval $(-\infty, 0)$. But polynomials are continuous everywhere, so $x = 0$ is the only possible discontinuity for $|x|$. Since $|0| = 0$, to prove the continuity at $x = 0$ we must show that

$$\lim_{x \to 0} |x| = 0 \qquad (1)$$

Because the formula for $|x|$ changes at 0, it will be helpful to consider the one-sided limits at 0 rather than the two-sided limit. We obtain

$$\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0 \quad \text{and} \quad \lim_{x \to 0^-} |x| = \lim_{x \to 0^-} (-x) = 0$$

Thus, (1) holds and $|x|$ is continuous at $x = 0$. ◀

2.5.4 THEOREM. *A rational function is continuous at every number where the denominator is nonzero.*

Example 3 For what values of x is there a hole or a gap in the graph of

$$y = \frac{x^2 - 9}{x^2 - 5x + 6}?$$

Solution. The function being graphed is a rational function, and hence is continuous at every number where the denominator is nonzero. Solving the equation

$$x^2 - 5x + 6 = 0$$

yields discontinuities at $x = 2$ and at $x = 3$. ◀

<u>Notes</u>: Undefined value: ∞^0 , $\infty - \infty$, $\infty * 0$, $\frac{\infty}{0}$, $\frac{\infty}{\infty}$, $\frac{0}{0}$, 0^0 .